Midterm examination
Business mathematics

20^{th} December 2005

Time: 3 hours (1–4pm)

1.

a. Given
$$f(x) = ax^2 + bx + c$$
where $a = c = 1$ and $b = 2$. Find all the x-intercepts, y-intercepts, and critical points.

b. Given
$$\ln y = 1 - x$$
Find y.

c. Given
$$\left(\frac{a^x a^y}{a^z}\right)^n = a^b$$
Find b.

d. Given
$$\log y = 3\log a + \log b - \log c$$
Find y.

[10]

Solution.

a.
$$f(x) = x^2 + 2x + 1$$
#

x-intercept, $y = 0$;
$$x^2 + 2x + 1 = 0$$
$$(x+1)(x+1) = 0$$
$$x = -1$$
$f(x)$ touches x-axis at $x = -1$
#

Critical point; $f'(x) = 0$;
$$2x + 2 = 0$$
$$x = -1$$
#

y-intercept, $x = 0$;
$$y = 1$$
#

b.
$$y = e^{1-x}$$
#

c.
$$a^{(x+y-z)n} = a^b$$
$$b = (x + y - z)n$$
#

d.
$$\log a^3 + \log b - \log c = \log y$$
$$\log \frac{a^3 b}{c} = \log y$$
$$y = \frac{a^3 b}{c}$$
#

2.

a.

$$q = \ln x + \ln y$$

b.

$$z = x^3 + x^2 + x + 2xy + xy^2$$

c.

$$p = 150e^{0.74t}$$

Find the first- and second-order partial derivatives.

[10]

Solution.

a.

$$\frac{\partial q}{\partial x} = \frac{1}{x}$$

#

$$\frac{\partial q}{\partial y} = \frac{1}{y}$$

#

$$\frac{\partial^2 q}{\partial x^2} = -\frac{1}{x^2}$$

#

$$\frac{\partial^2 q}{\partial x \partial y} = 0$$

#

$$\frac{\partial^2 q}{\partial y^2} = -\frac{1}{y^2}$$

#

$$\frac{\partial^2 q}{\partial y \partial x} = 0$$

#

b.

$$\frac{\partial z}{\partial x} = 3x^2 + 2x + 1 + 2y + y^2$$

#

$$\frac{\partial z}{\partial y} = 2x + 2xy$$

#

$$\frac{\partial^2 z}{\partial x^2} = 6x + 2$$

#

$$\frac{\partial^2 z}{\partial x \partial y} = 2 + 2y$$

#

$$\frac{\partial^2 z}{\partial y^2} = 2x$$

#

$$\frac{\partial^2 z}{\partial y \partial x} = 2 + 2y$$

#

c.

$$\frac{\partial p}{\partial t} = 150(0.74)e^{0.74t} = 111e^{0.74t}$$

#

$$\frac{\partial^2 p}{\partial t^2} = 111(0.74)e^{0.74t} = 82.14e^{0.74t}$$

#

3. Find the determinant of the following matrices.

a.
$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$
[2]

b.
$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$
[3]

c.
$$\begin{bmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \\ m & n & o & p \end{bmatrix}$$
[5]

Solution.

a.
$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$
#

b.
$$\begin{aligned} \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} &= a\begin{vmatrix} c & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix} \\ &= a(ci - fh) - b(di - fg) + c(dh - eg) \\ &= aci - afh - bdi + bfg + cdh - ceg \end{aligned}$$
#

c.
$$\begin{aligned} \begin{vmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \\ m & n & o & p \end{vmatrix} &= a\begin{vmatrix} f & g & h \\ j & k & l \\ n & o & p \end{vmatrix} - b\begin{vmatrix} e & g & h \\ i & k & l \\ m & o & p \end{vmatrix} + c\begin{vmatrix} e & f & h \\ i & j & l \\ m & n & p \end{vmatrix} - d\begin{vmatrix} e & f & g \\ i & j & k \\ m & n & o \end{vmatrix} \\ &= a\left(f(kp - lo) - g(jp - ln) + h(jo - kn)\right) \\ &\quad - b\left(e(kp - lo) - g(ip - lm) + h(io - km)\right) \\ &\quad + c\left(e(jp - ln) - f(ip - lm) + h(in - jm)\right) \\ &\quad - d\left(e(jo - kn) - f(io - km) + g(in - jm)\right) \\ &= afkp - aflo - agjp + agln + ahjo - ahkn \\ &\quad - bekp + belo + bgip - bglm - bhio + bhkm \\ &\quad + cejp - celn - cfip + cflm + chin - chjm \\ &\quad - dejo + dekn + dfio - dfkm - dgin + dgjm \end{aligned}$$
#

4. The relationship between the total revenue r_t, the price p, and the output quantity q is

$$r_t = pq$$

The demand function is $p = a - bq$, where a and b are positive constants.
Find r_t, the marginal revenue r_m, and the average revenue r_a. Then find q at the maximum r_t. And then sketch the graphs of r_t, r_m and r_a. (assume $a_2 > 4b$)

[10]

Solution.

$$r_t = pq = (a - bq)q = aq - bq^2$$

#

$$r_m = \frac{d\,r_t}{d\,q} = a - 2bq$$

#

$$r_a = \frac{r_t}{q} = p = a - bq$$

#

At maximum r_t;

$$r_t' = a - 2bq = 0$$
$$q = \frac{a}{2b}$$

#

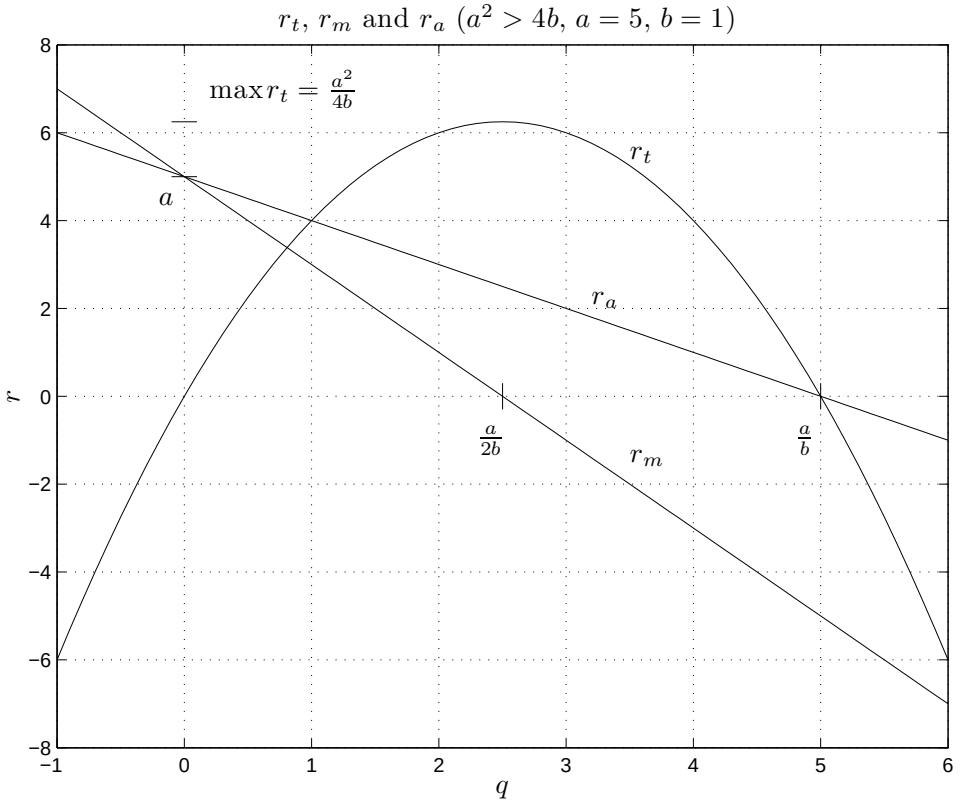

Figure 1

$$q = \frac{a}{2b}; \quad r_t = a\left(\frac{a}{2b}\right) - b\left(\frac{a^2}{4b^2}\right) = \frac{a^2}{4b^2}\left(1 - \frac{b}{2b}\right) = \frac{a^2}{4b}$$

$$q = 0; \quad r_t = 0$$

$r_t = 0$;

$$q(a - bq) = 0$$
$$q = 0, \quad \frac{a}{b}$$
$$q = 0; \quad r_m = a$$
$$r_m = 0; \quad q = \frac{a}{2b}$$
$$q = 0; \quad r_a = a$$
$$r_a = 0; \quad q = \frac{a}{b}$$

This can be summarised as Figure 1.

5.

a. Let A , B , and C represent the second-order conditions of critical point of function. Suppose the graph of a function has the shape as shown in Figure 2.

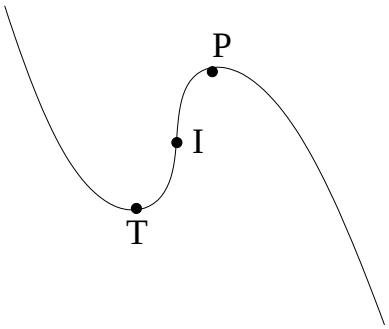

Figure 2

Which of these conditions is satisfied at points I, P and T?[3] Explain.[2]

b.

$$f(x) = \frac{3}{5}x^5 - \frac{9}{4}x^4 + x^3 + \frac{9}{2}x^2 - 6x + 7$$

Find $f'(x)$ and $f''(x)$.[1] Show that 1, -1 and 2 are the critical points.[2] Which of these are maximum, minimum or inflection point?[2]

Solution.

a.

$$\left.\begin{array}{ll} I & \leftarrow B \\ p & \leftarrow C \\ T & \leftarrow A \end{array}\right\}$$

#

At I, P and T, $f'(\cdot) = 0$.

#

At I, P and T, $f''(\cdot) = 0$, < 0, and > 0 respectively.

#

b.

$$\left.\begin{array}{ll} f'(x) & = 3x^4 - 9x^3 + 3x^2 + 9x - 6 \\ f''(x) & = 12x^3 - 27x^2 + 6x + 9 \end{array}\right\}$$

#

$$f'(x) = \left(3x^2 - 6x + 3\right)(x+1)(x-2) = 3(x-1)^2(x+1)(x-2)$$

Critical points, 1, -1 and 2. At these points $f'(\cdot) = 0$.

#

$$\left.\begin{array}{lll} f''(\cdot) = 0 & \rightarrow 1 & \text{inflection point} \\ f''(-1) < 0 & \rightarrow -1 & \text{maximum point} \\ f''(2) > 0 & \rightarrow 2 & \text{minimum point} \end{array}\right\}$$

#

6. Solve the following programme by the simplex method.

$$\begin{aligned} \text{maximise:} \quad & z = 3x_1 + 4x_2 + 5x_3 \\ \text{subject to:} \quad & x_1 + x_2 + x_3 \leq 2 \\ & x_1 + x_2 + 3x_3 \leq 1 \\ & 3x_1 + 2x_2 + x_3 \leq 4 \\ \text{with:} \quad & \text{all the variables non-negative} \end{aligned}$$

[10]

Solution. Draw simplex tables.

		x_1	x_2	x_3	x_4	x_5	x_6	
		3	4	5	0	0	0	
x_4	0	1	1	1	1	0	0	2
x_5	0	1	1	3	0	1	0	1
x_6	0	3	2	1	0	0	1	4
		-3	-4	-5	0	0	0	0

	x_1	x_2	x_3	x_4	x_5	x_6	
x_4	$\frac{2}{3}$	$\frac{2}{3}$	0	1	$-\frac{1}{3}$	0	$\frac{5}{3}$
x_3	$\frac{1}{3}$	$\frac{1}{3}$	1	0	$\frac{1}{3}$	0	$\frac{1}{3}$
x_6	$\frac{8}{3}$	$\frac{5}{3}$	0	0	$-\frac{1}{3}$	1	$\frac{11}{3}$
	$-\frac{4}{3}$	$-\frac{7}{3}$	0	0	$\frac{5}{3}$	0	$\frac{5}{3}$

	x_1	x_2	x_3	x_4	x_5	x_6	
x_4	0	0	-2	1	-1	0	1
x_2	1	1	3	0	1	0	1
x_6	1	0	-5	0	$\frac{4}{3}$	1	2
	1	0	7	0	4	0	4

$x_4^* = 1$, $x_2^* = 1$, $x_6^* = 2$, $x_1^* = x_3^* = x_5^* = 0$ and $z^* = 4$.

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